Math 146C - Ordinary and Partial Differential Equations III
Quiz 4
May 12, 2011

Name: Key

Total
/10

Problem 1. *(10 points) Let $f : \mathbb{C} \to \mathbb{C}$ be a complex analytic function given by $f(x,y) = u(x,y) + iv(x,y)$ (thinking as a complex number $z = x + iy = (x,y)$) so that u and v are $C^2(\mathbb{R}^2)$. Since f is complex analytic, we have that u and v satisfy the* **Cauchy-Riemann equations***:*

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Show that both u and v are harmonic, i.e. satisfy the Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u = 0$$

Similarly

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \Delta v = 0.$$